REVIEW

Vectors

Addition, ~~sc~~ mult by scalar

Dot prod, X prod

$A \cdot (B \times C)$

$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$

$\nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ Uphill

$\nabla \cdot \underline{A}$, $\underline{V} = \underline{r} f(r) \Rightarrow \nabla \cdot \underline{V} = 3 f(r) + r \frac{\partial f}{\partial r}$

$\nabla \times \underline{A}$

Gauss thm: $\int_S \underline{A} \cdot d\underline{\sigma} = \int_V \nabla \cdot \underline{A} \, d\tau$

Stokes thm: $\oint_C \underline{A} \cdot d\underline{r} = \int_S (\nabla \times \underline{A}) \cdot d\underline{\sigma}$

Potential thy: $\underline{F} = -\nabla \phi \Leftrightarrow \nabla \times \underline{F} = \underline{0} \Leftrightarrow \oint_C E \cdot d\underline{r} = 0$

Gauss' law: Let $\underline{E}(\underline{r}) = \frac{\alpha}{r^2}\hat{r}$ $\left\{ \alpha = \frac{q}{4\pi\varepsilon_0} \text{ or } -\frac{GM}{r^2} \right.$

$\Rightarrow \int_S \underline{E} \cdot d\underline{\sigma} = \int_V \frac{\rho}{\varepsilon_0} d\tau = \frac{Q}{\varepsilon_0}$

$\Updownarrow$

$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$

Dirac δ: $\delta(x) = 0 \quad \forall x \neq 0$

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\,dx = f(0)$$

Curvilinear coord systems. $\{q_i\}$

Scale factors: $ds_i = h_i \, dq_i$

Cyl. coords: $\rho, \phi, z \qquad h_\rho = 1, \; h_\phi = \rho, \; h_z = 1$

Spherical: $r, \vartheta, \phi \qquad h_r = 1, \; h_\vartheta = r, \; h_\phi = r \sin\vartheta$

$$\nabla \psi = \sum_j \hat{e}_j \frac{1}{h_j} \frac{\partial \psi}{\partial q_j}$$

$$\nabla \cdot \underline{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (V_1 h_2 h_3) + c.p. \right]$$

$$\nabla \times \underline{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ V_1 h_1 & V_2 h_2 & V_3 h_3 \end{vmatrix}$$

Determinants + matrices

$m \times n$ matrix: $\begin{array}{cc} & n \text{ cols} \rightarrow \\ m \text{ rows} \downarrow & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix} \end{array} = \underline{\underline{A}} = (a_{ij})$

$m \times 1$: Col vector of dim m: $\begin{pmatrix} a_{1} \\ \vdots \\ a_m \end{pmatrix} = |a\rangle$

$1 \times n$: row vector of dim n: $(a_1, \ldots, a_n) = \langle a|$

Addition, mult by scalar: term by term.

Matrix multiplication:

$$(\underline{\underline{A}}\,\underline{\underline{B}})_{ij} = \sum_k a_{ik} b_{kj}\,, \quad AB \neq BA \text{ in general}$$

$n \times n$!

Hom. set of linear eqns: $\underline{\underline{A}}\,|v\rangle = 0$

A $|v\rangle \neq 0$ can be found only if

$\det \underline{\underline{A}} \equiv |\underline{\underline{A}}| = 0$ (i.e. if A is singular)

determinant:

$$|\underline{\underline{A}}| = \sum_{\alpha\beta\gamma\omega} \varepsilon_{\alpha\beta\gamma\omega} a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

$$\varepsilon_{\alpha\beta\gamma\ldots} = \begin{cases} +1, & \alpha\beta\gamma\ldots \text{ even perm of } 1,\ldots,n \\ -1, & \text{odd} \\ 0, & \text{repeated index.} \end{cases}$$

Expansion by minors

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(\ ) + a_{13}(\ )$$

Properties of det's:

- $|\tilde{A}| = |A|$
- det changes sign if 2 cols or 2 rows interchanged
  $\Rightarrow$ det $= 0$ if 2 cols or 2 rows $=$
- det $= 0$ if one row/col $= 0$
- linear in row/col
- Unchanged if ~~one~~ mult of one r/c added to another
- $|\mathbb{1}| = 1$

Inverse: $A^{-1}A = AA^{-1} = \mathbb{1}$ possible only if $|A| \neq 0$.

$\perp$ matrix: $\tilde{A} = A^{-1}$
Unitary: $A^{+} = A^{-1}$

Symm. $\tilde{A} = A$
Herm $A^{+} = A$

Sim. transform: $A' = BAB^{-1}$
most interesting if B $\perp$ or unitary

Rules: $\widetilde{AB} = \tilde{B}\tilde{A}$, $(AB)^\dagger = B^\dagger A^\dagger$

Eigenvalues / eigenvectors:

$$A|\psi\rangle = \lambda|\psi\rangle$$
$$\Updownarrow$$
$$(A - \lambda \mathbb{1})|\psi\rangle = 0 \quad \text{hom lin eqns}$$

$\Rightarrow$ $|A - \lambda \mathbb{1}| = 0$ secular / characteristic eqn.

For symm or herm matrices, $\lambda \in \mathbb{R}$.

$\langle \lambda_\alpha | \lambda_\beta \rangle = \delta_{\alpha\beta}$ if $\lambda_\alpha \neq \lambda_\beta$. orthonormality

Sequences / series.

Finite series: $S_i = \sum_{n=1}^{i} u_i$, ∞ : $S_\infty = \sum_{n=1}^{\infty} u_i$ if $\exists$.

Convergence, divergence

$\lim_{i\to\infty} S_i$ finite.

Geometric series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x|<1$

for $u_i > 0$.

Convergence tests: Comparison
Root, Ratio, $\int$

Alt series: Leibnitz: conv. if $|u_{n+1}| \leq |u_n|$
and $\lim_{n\to\infty} |u_n| = 0$: conv.
Error < first neglected term.

Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$ if convergent.

e.g.: $e^x = \sum \frac{x^n}{n!}$

Uniform conv for series of functions
Weierstrass M-test. Can be integrated and differentiated term wise

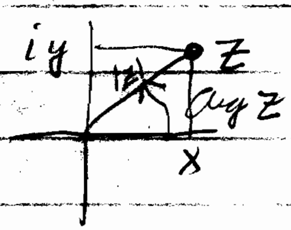
Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Complex analysis:



$$z = x + iy = |z|\,e^{i\,\mathrm{Arg}\,z}$$

$$\xi = \alpha + i\beta = |\xi|\,e^{i\,\mathrm{Arg}\,\xi}$$

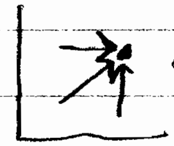
$$z\xi = |z|\cdot|\xi|\,e^{i(\mathrm{Arg}\,z + \mathrm{Arg}\,\xi)}$$

$$z^* = x - iy = |z|\,e^{-i\,\mathrm{Arg}\,z}$$

$$e^{in\phi} = (\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$$

$$e^{i\pi} = -1$$

Differentiation of complex functions



all directions =

$f(z) = u(x,y) + i\,v(x,y)$

CRC: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Diff'able in neighborhood of $z \Leftrightarrow$ anal. at z.

Complex $\int$: $\displaystyle\int_{z_1}^{z_2} f(z)\,dz = \int_{x_1,y_1}^{x_2,y_2} (u(x,y) + i\,v(x,y))(dx + i\,dy)$

Cauchy's $\int$ thm: $f(z)$ anal on + inside closed path C, then
$$\oint_C f(z)\,dz = 0.$$

Cauchy's $\int$ formula:
$$\frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}\,dz = \begin{cases} f(z_0), & z_0 \text{ inside } C \\ 0, & z_0 \text{ outside} \end{cases}$$

Diff eqns.

ODE : contains only od's
PDE : contains $\frac{\partial}{\partial}$

Linear : No squares or powers in unknown funct. or ders.

Inhom DE : Contains a term indep of unknown funct.
Hom DE : Contains no such term

Sep. of vars.: e.g.: $\frac{dy}{dx} = -\lambda y$

$$\Downarrow$$

$$\frac{dy}{y} = -\lambda\, dx$$

Sol'n of inhom. DE : $y(x) = y_h(x) + y_p(x)$

Linear ODE w/ const coeffs:

$$a_n \frac{d^n y}{dx^n} + \dots + a_0 y = F(x)$$

Hom eqn. solved by e^{sx} $\Rightarrow$ n-th order eqn for s.

Harm. osc.

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \omega_0^2 y = F(t)$$

$\neq$ sol'ns depending on $\alpha^2 - \omega_0^2$.

Underdamped ($\alpha^2 < \omega_0^2$): , $\omega = \sqrt{\omega_0^2 - \alpha^2}$

$$y(t) = e^{-\alpha t}\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

$$= Ce^{-\alpha t}\cos(\omega t + \delta)$$

Eqn w/ x-dep coeffs:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Also solve by sep of vars + var. of coeff.

PDEs:

Wave: $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Diff'n: $$D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Sep of vars: $$u = \alpha(x)\,\beta(y)$$

e.g.: $$\frac{\alpha''}{\alpha} = \frac{\beta''}{c^2\beta} = \underbrace{\gamma}_{\text{sep'n const.}}$$