Physics 5492
Condensed Matter Physics II
Problem Set 2
Due: Tuesday, Feb 3, 2009

2.1 **Plasmon Dispersion.**
Consider the Lindhard expression for the dielectric function $\epsilon(\vec{q},\omega)$ of a free electron gas in the limit $\omega \gg v_F q$ and show that it vanishes when

$$\omega = \omega_p \left(1 + \frac{3}{10}\frac{v_F^2}{\omega_p^2}q^2 + O(q^4)\right)$$

where v_F is the Fermi velocity and ω_p is the plasma frequency. Comment on the significance of this result.

2.2 **Friedel Oscillations.**
Consider the $\omega = 0$ dielectric function

$$\epsilon(\mathbf{q}) = 1 - \frac{4\pi}{q^2}\chi(\mathbf{q},0)$$

We found in class that χ is roughly constant at small q, drops sharply at $q = 2k_F$ and then goes to zero as q goes to infinity.

(a) Using the following simplified model for χ,

$$\chi(\mathbf{q},0) = \begin{cases} -k_0^2/4\pi & \text{for } q < 2k_F \\ 0 & \text{for } q > 2k_F, \end{cases}$$

i.e. Thomas-Fermi for $q < 2k_F$ and no screening ($\epsilon = 1$) for $q > 2k_F$, show that given a point charge Q for which the external potential is $\phi^{\text{ext}}(r) = Q/r$ that the screened potential is given by the following integral

$$\phi(r) = \frac{2}{\pi}\frac{Q}{r}\int_0^\infty F(q)\sin qr \; dq$$

where

$$F(q) = \begin{cases} \frac{q}{q^2+k_0^2} & \text{for } q < 2k_F \\ \frac{1}{q} & \text{for } q > 2k_F. \end{cases}$$

(b) Show that the discontinuity in $F(q)$ at $q = 2k_F$ leads to Friedel oscillations in $\phi(r)$ for $k_F r \gg 1$ and find an analytic expression for these oscillations valid in this limit.

(c) Discuss how your answer for (b) compares with the result given in class for Friedel oscillations in the three dimensional electron gas,

$$\phi(r) \sim \frac{1}{r^3}\cos 2k_F r,$$

and account for any differences.

2.3 Problem 2, Chapter 26 of A&M, Pg. 529.

2.4 **Compressibility of a Fermi Liquid.** (**Optional**)
The low energy excitations of a Fermi liquid are labeled by quasiparticle occupation numbers $n_{\mathbf{k}\sigma}$, or, more precisely, $\delta n_{\mathbf{k}\sigma}$, the *deviation* of the occupation numbers from their ground state values $n^0_{\mathbf{k}\sigma}$,

$$\delta n_{\mathbf{k}\sigma} = n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}\sigma}.$$

For a given $\delta n_{\mathbf{k}\sigma}$ the energy of the system is then given by the Landau energy functional,

$$E[\delta n_{\mathbf{k}\sigma}] = E_0 + \sum_{\mathbf{k}\sigma} E_\sigma(\mathbf{k})\delta n_{\mathbf{k}\sigma} + \frac{1}{2V}\sum_{\mathbf{k}\sigma;\mathbf{k}'\sigma'} f(\mathbf{k},\sigma;\mathbf{k}',\sigma')\delta n_{\mathbf{k}\sigma}\delta n_{\mathbf{k}'\sigma'}.$$

The energy to create an excitation (by switching $n_{\mathbf{k}\sigma}$ for a particular $\mathbf{k}$ and σ from 0 to 1, or from 1 to 0), is then easily shown to be

$$E^T_\sigma(\mathbf{k}) = E_\sigma(\mathbf{k}) + \frac{1}{V}\sum_{\mathbf{k}'\sigma'} f(\mathbf{k},\sigma;\mathbf{k}',\sigma')\delta n_{\mathbf{k}'\sigma'}. \tag{1}$$

The first term in Eq. (1) is the renormalized quasiparticle dispersion, characterized by the effective mass parameter

$$m^* = \frac{\hbar k_F}{v_F} = \frac{\hbar^2 k_F}{|\nabla_k E_\sigma(\mathbf{k})_{k=k_F}|}.$$

The second term in Eq. (1) is the contribution to the quasiparticle energy due to its interaction with the other excited quasiparticles in the system (characterized by the Landau f-function, $f(\mathbf{k}\sigma,\mathbf{k}'\sigma')$. For simplicity lets consider an isotropic Fermi liquid, e.g. liquid He^3.

It is possible to express the compressibility of a given system as follows

$$\kappa = \frac{V}{N^2}\left(\frac{\partial N}{\partial \mu}\right)_V.$$

Thus, to find the compressibility of a Fermi liquid we need to determine $(\partial N/\partial\mu)_V$ for that liquid. Here N is the total number of particles.

(a) At $T = 0$, let $n_{\mathbf{k}\sigma}$ denote the occupation numbers when the chemical potential is μ, and $n^0_{\mathbf{k}\sigma}$ denote the occupation numbers in the ground state when the chemical potential is E_F. Show that the change in the occupation numbers when the chemical potenital is changed from E_F to μ is

$$\delta n_{\mathbf{k}\sigma} = \Theta(\mu - E^T_\sigma(\mathbf{k})) - \Theta(E_F - E_\sigma(\mathbf{k}))$$

where $E^T_\sigma(\mathbf{k})$ is given by Eq. (1) (and so depends self-consistently on $\delta n_{\mathbf{k}\sigma}$).

(b) Show that

$$\frac{\partial N}{\partial \mu} = \sum_{\mathbf{k}\sigma} \frac{\partial \delta n_{\mathbf{k}\sigma}}{\partial \mu} \tag{2}$$

where

$$\frac{\partial \delta n_{\mathbf{k}\sigma}}{\partial \mu} = \delta(\mu - E^T_\sigma(\mathbf{k})) \left(1 - \frac{1}{V}\sum_{\mathbf{k}'\sigma'} f(\mathbf{k}\sigma;\mathbf{k}'\sigma')\frac{\partial \delta n_{\mathbf{k}'\sigma'}}{\partial \mu}\right). \tag{3}$$

(c) From Eq. (3) show that for an isotropic system

$$\sum_{\mathbf{k}'\sigma'} f(\mathbf{k}\sigma;\mathbf{k}'\sigma')\frac{\partial \delta n_{\mathbf{k}'\sigma'}}{\partial \mu} = \frac{F_0^s}{1+F_0^s} \tag{4}$$

where

$$F_0^s = \int \frac{d^3k}{(2\pi)^3}\delta(E_\sigma(\mathbf{k}) - \mu)(f(\mathbf{k}\uparrow, \mathbf{k}'\uparrow) + f(\mathbf{k}\uparrow, \mathbf{k}'\downarrow))$$

is the $l = 0$ symmetric Fermi liquid parameter (you may want to check that this definition is the same as that given in class).

(d) Finally, from Eq. (4) and Eq. (2), show that

$$\frac{\partial N}{\partial \mu} = V g^*(E_F)\frac{1}{1+F_0^s}$$

where $g^*(E_F) = \frac{1}{V}\sum_{\mathbf{k}} \delta(E_\sigma(\mathbf{k}) - E_F)$ is the density of states at the Fermi level for the renormalized quasiparticle band $E_\sigma(\mathbf{k})$. Thus we see that, due to quasiparticle interactions, $\partial N/\partial \mu$ and hence also the compressibility of a Fermi liquid are renormalized by a factor of $1/(1 + F_0^s)$.