Physics 5492
Condensed Matter Physics II
Problem Set 3
Due: Thu, Feb 19, 2009

3.1 Problem 3, Chapter 26 of A&M, Pg. 530.

3.2 **Overscreening in Real Space and Time.**
In class we found the following approximate expression for the dielectric constant of a metal which includes the screening effects of both the fast electrons and the slow acoustic phonons:

$$\epsilon(\mathbf{q},\omega) = \left(1 + \frac{k_0^2}{q^2}\right)\left(1 - \frac{\omega_{\mathbf{q}}^2}{\omega^2}\right) \qquad \text{where} \quad \omega_{\mathbf{q}} = v_s q.$$

Here v_s is the velocity of sound and k_0 is the Thomas-Fermi wavevector. Consider the following time-dependent external potential

$$\phi^{\text{ext}}(\mathbf{r},t) = \begin{cases} -\frac{e}{r}e^{\alpha t} & t < 0 \\ 0 & t > 0 \end{cases}$$

where α is a positive number which is taken to zero at the end of the calculation. (Physically the $\exp \alpha t$ is there because we imagine adiabatically turning on the potential starting at $t = -\infty$. Mathematically you'll need it to keep some of the integrals you'll have to do well behaved.)

(a) Find the Fourier transform of ϕ^{ext}

$$\phi^{\text{ext}}(\mathbf{q},\omega) = \int d^3r \int dt \; \phi^{\text{ext}}(\mathbf{r},t) \; e^{i\mathbf{q}\cdot\mathbf{r}} \; e^{i\omega t}.$$

(b) Compute the screened potential as a function of $\mathbf{r}$ and t by performing the integral

$$\phi(\mathbf{r},t) = \int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi} \; \frac{\phi^{\text{ext}}(\mathbf{q},\omega)}{\epsilon(\mathbf{q},\omega)} \; e^{-i\mathbf{q}\cdot\mathbf{r}} \; e^{-i\omega t}.$$

Note: In doing this integral assume that the zeros in ϵ, and hence the poles in the integrand, occur just below the real frequency axis.

(c) You have just calculated the screened potential seen by electrons in a metal due to an electron which is fixed at $r = 0$ starting at time $t = -\infty$ and then suddenly removed at time $t = 0$. Plot your result as a function of r for various times t to show that the electron leaves behind regions where the potential is *attractive* for other electrons. Show where these attractive regions occur in space and time and discuss their significance.

3.3 Canonical Transformation. The Hamiltonian describing interacting electrons and phonons in a metal can be written as

$$H = H_0 + H_I,$$

where the first term,

$$H_0 = \sum_{\mathbf{k},\sigma} E_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} (a^\dagger_{\mathbf{q}} a_{\mathbf{q}} + 1/2),$$

describes free electrons and phonons, and the second term,

$$H_I = \sum_{\mathbf{k},\mathbf{q},\sigma} M_{\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma} (a_{\mathbf{q}} + a^\dagger_{-\mathbf{q}}),$$

describes the electron-phonon interaction.

The phonon mediated electron-electron interaction can be derived systematically for this system by performing a **canonical transformation** on the Hamiltonian H. Such a transformation is generated by an operator S with the property that $S^\dagger = -S$. The operator $U = e^{-S}$ is then unitary, i.e. $U^\dagger = U^{-1}$, and applying this unitary transformation to the Hamiltonian H yields

$$\begin{aligned} H_S &= e^{-S} H e^S \\ &= H + [H, S] + \frac{1}{2}[[H, S], S] + \cdots \\ &= H_0 + (H_I + [H_0, S]) + \frac{1}{2}[(H_I + [H_0, S]), S] + \frac{1}{2}[H_I, S] + \text{higher order terms.} \end{aligned}$$

Here, in the final expressions, it is assumed that S and H_I are both ‘small’ and of the same order, and the terms are grouped accordingly.

(a) Show that if we take the operator S to be

$$S = \sum_{\mathbf{k},\mathbf{q},\sigma} M_{\mathbf{q}} (\alpha_{\mathbf{qk}} a^\dagger_{-\mathbf{q}} + \beta_{\mathbf{qk}} a_{\mathbf{q}}) c^\dagger_{\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k}\sigma}$$

where

$$\alpha_{\mathbf{qk}} = \frac{1}{E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}} - \hbar\omega_{\mathbf{q}}} \quad \text{and} \quad \beta_{\mathbf{qk}} = \frac{1}{E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}} + \hbar\omega_{\mathbf{q}}}$$

then

$$H_I + [H_0, S] = 0$$

and the transformed Hamiltonian becomes

$$H_S = H_0 + \frac{1}{2}[H_I, S] + \text{higher order terms.}$$

(b) Evaluate $\frac{1}{2}[H_I, S]$ and show that it leads to a term of the form

$$H_{el-el} = \sum_{\substack{\mathbf{k},\mathbf{k}',\mathbf{q} \\ \sigma,\sigma'}} \frac{2|M_{\mathbf{q}}|^2 \hbar\omega_{\mathbf{q}}}{(E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}})^2 - (\hbar\omega_{\mathbf{q}})^2} c^\dagger_{\mathbf{k}+\mathbf{q},\sigma} c^\dagger_{\mathbf{k}'-\mathbf{q},\sigma'} c_{\mathbf{k}'\sigma'} c_{\mathbf{k}\sigma}$$

as well as terms proportional to $c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$ which can be grouped with the one-electron energy in H_0.