Physics 5492
Condensed Matter Physics II
Problem Set 4

Due: Thu, Mar 5, 2009

4.1 In class we obtained the Fourier components, $a_{\mathbf{k}}$, of a $\mathbf{Q} = 0$ Cooper pair wave function,

$$a_{\mathbf{k}} = \begin{cases} C\frac{1}{E-2E_{\mathbf{k}}} & k_f < |\mathbf{k}| < k_a \\ 0 & \text{otherwise.} \end{cases}$$

Here C is a normalization constant,

$$\frac{\hbar^2 k_f^2}{2m} = E_f \qquad \text{and} \qquad \frac{\hbar^2 k_a^2}{2m} = E_f + \hbar\omega_D,$$

where E_f is the Fermi energy and $\hbar\omega_D$ is the Debye energy, and $E = 2E_f - \Delta$ where Δ is the pair binding energy.

(a) Using this result, evaluate the expectation value $\langle r^2 \rangle$ for a Cooper pair where r is the relative coordinate of the two electrons. Justify any approximations you make in your calculation and express your answer in terms of Δ and the Fermi velocity v_f.

(b) Let $\xi = \sqrt{\langle r^2 \rangle}$ be the root-mean-square size of a Cooper pair. Pick your favorite superconducting element from Table 34.2 in Ashcroft and Mermin and compute ξ for that element.

4.2 Consider the BCS wave function

$$|\psi_{\rm BCS}\rangle = \prod_{\mathbf{k}} [u_{\mathbf{k}} + v_{\mathbf{k}} b_{\mathbf{k}}^\dagger]|0\rangle$$

where

$$b_{\mathbf{k}}^\dagger = c_{\mathbf{k},\uparrow}^\dagger c_{-\mathbf{k},\downarrow}^\dagger$$

is a pair creation operator.

(a) Show that,

$$u_{\mathbf{k}} + v_{\mathbf{k}} b_{\mathbf{k}}^\dagger = u_{\mathbf{k}} \exp\left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger\right)$$

where $g_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}}$. (Hint: Taylor expand the exponential and use the anticommutation relations of the Fermi creation operators to show that all but the first two terms vanish).

(b) Using this result show that up to a normalization constant

$$|\psi_{\rm BCS}\rangle = \exp\left(\sum_{\mathbf{k}} g_{\mathbf{k}} b_{\mathbf{k}}^\dagger\right)|0\rangle.$$

(c) Let P_N be a projection operator which projects out states for which the total number of electrons is N, where N is an even number. Using your result from (b) show that, again up to a normalization constant,

$$P_N|\psi_{\rm BCS}\rangle = \left(\sum_{\mathbf{k}} g_{\mathbf{k}} b^\dagger_{\mathbf{k}}\right)^{N/2} |0\rangle$$

and interpret this result.

4.3 Consider a peculiar metal described by the reduced BCS Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

where $\xi_{\mathbf{k}}$ is the normal state electron energy dispersion measured from the Fermi energy

$$\xi_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - E_F.$$

Let

$$V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -\frac{v_0}{\Omega}\frac{|\xi_{\mathbf{k}}||\xi_{\mathbf{k}'}|}{(\hbar\omega_D)^2} & \text{for } |\xi_{\mathbf{k}}| < \hbar\omega_D \text{ and } |\xi_{\mathbf{k}'}| < \hbar\omega_D \\ 0 & \text{otherwise} \end{cases}$$

where $v_0 > 0$ and Ω is the volume of the system.

(a) Solve the $T = 0$ BCS gap equation for this system, obtain $\Delta_{\mathbf{k}}$, and give a criterion for the existence of superconductivity. (This will be a condition on $N(0)v_0$ required for the gap equation to have a solution).

(b) Sketch the density of excited states as a function of energy out to and beyond $\hbar\omega_D$.