Physics 5492
Condensed Matter Physics II
Problem Set 5
Due: Thu, April 2, 2009

5.1 The Bogoliubov transformation has the form

$$\begin{aligned}\gamma_{\mathbf{k}0} &= u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} \\ \gamma_{\mathbf{k}1} &= u_{\mathbf{k}} c_{-\mathbf{k}\downarrow} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} \\ \gamma^\dagger_{\mathbf{k}0} &= u^*_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} \\ \gamma^\dagger_{\mathbf{k}1} &= u^*_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} + v^*_{\mathbf{k}} c_{\mathbf{k}\uparrow}\end{aligned}$$

where $c^\dagger_{\mathbf{k}\sigma}$ and $c_{\mathbf{k}\sigma}$ are creation and annihilation operators for plane wave states (or Bloch states) with wave vector $\mathbf{k}$ and spin σ.

(a) Show that if $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$ then the γ operators satisfy the usual Fermi anticommutation relations

$$\{\gamma_{\mathbf{k}\mu}, \gamma^\dagger_{\mathbf{k}',\mu'}\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mu,\mu'}, \qquad \{\gamma^\dagger_{\mathbf{k}\mu}, \gamma^\dagger_{\mathbf{k}'\mu'}\} = 0, \quad \text{and} \quad \{\gamma_{\mathbf{k}\mu}, \gamma_{\mathbf{k}'\mu'}\} = 0.$$

(b) Verify the inverse transformation

$$\begin{aligned}c_{\mathbf{k}\uparrow} &= u^*_{\mathbf{k}}\gamma_{\mathbf{k}0} + v_{\mathbf{k}}\gamma^\dagger_{\mathbf{k}1} \\ c_{-\mathbf{k}\downarrow} &= -v_{\mathbf{k}}\gamma^\dagger_{\mathbf{k}0} + u^*_{\mathbf{k}}\gamma_{\mathbf{k}1} \\ c^\dagger_{\mathbf{k}\uparrow} &= u_{\mathbf{k}}\gamma^\dagger_{\mathbf{k}0} + v^*_{\mathbf{k}}\gamma_{\mathbf{k}1} \\ c^\dagger_{-\mathbf{k}\downarrow} &= -v^*_{\mathbf{k}}\gamma_{\mathbf{k}0} + u_{\mathbf{k}}\gamma^\dagger_{\mathbf{k}1}.\end{aligned}$$

(c) Apply this transformation to Eq. (3.41) in Tinkham and verify Eq. (3.43).

(d) Show that if

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \qquad \text{and} \qquad |v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$

then Eq. (3.45) is true. In doing this you may assume (though it is not necessary) that $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are real.

5.2 Consider the Ginzburg-Landau expression for the free energy density

$$F_s - F_n = \alpha|\psi(\mathbf{r})|^2 + \frac{\beta}{2}|\psi(\mathbf{r})|^4 + \left|\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A}\right)\psi(\mathbf{r})\right|^2.$$

(a) Show that if $\psi(\mathbf{r})$ is taken to be constant and $\alpha < 0$ and $\beta > 0$ then the free energy is minimized when $\psi = \psi_0$ where

$$|\psi_0|^2 = -\frac{\alpha}{\beta}$$

and the corresponding minimum free energy density is

$$F_s - F_n = -\frac{\alpha^2}{2\beta}.$$

(b) By setting the free energy gain obtained in (a) equal to the energy density of a uniform magnetic field obtain an expression for the thermodynamic critical field H_c. Then, using the fact that the London penetration depth λ and the Ginzburg-Landau coherence length ξ are defined by the relations

$$\frac{1}{\lambda^2} = \frac{4\pi e^{*2}|\psi_0|^2}{m^* c^2} \qquad \text{and} \qquad \xi^2 = \frac{\hbar^2}{2m^*|\alpha|}$$

derive the following relationship between H_c, ξ and λ,

$$H_c = \frac{\Phi_0}{2\sqrt{2}\pi\xi\lambda}.$$

Recall that minimizing the Ginzburg-Landau free energy leads to the following equation for the order parameter,

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A}\right)^2 \psi + \alpha\psi + \beta|\psi|^2\psi = 0. \tag{1}$$

Imagine that a superconductor described by this equation is placed in a uniform magnetic field $\mathbf{H} = \nabla \times \mathbf{A}$.

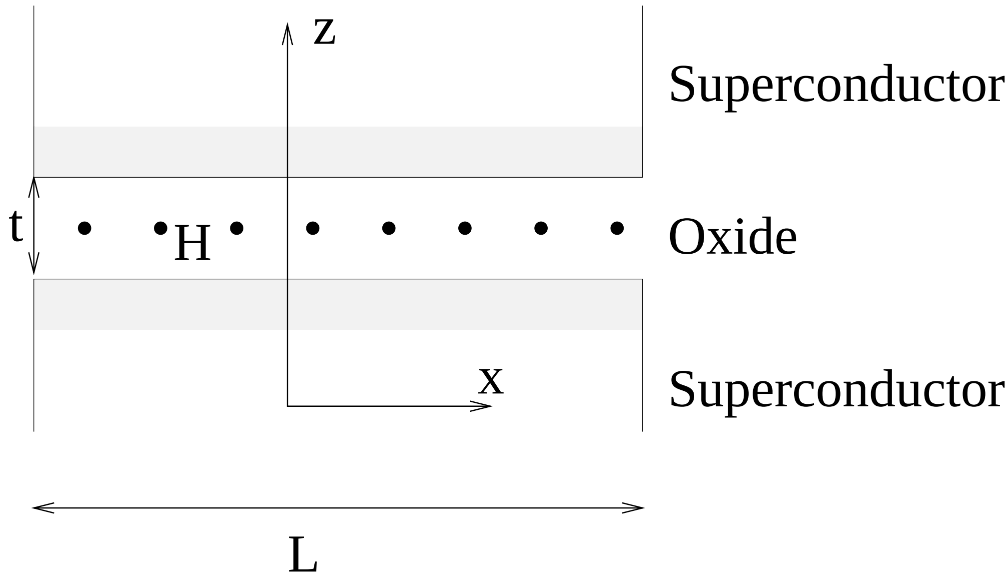
(c) Assume that $|\psi|/|\psi_0| \ll 1$ and linearize (1). Show that the resulting equation for ψ is the usual Schrödinger equation for a three-dimensional charged particle in a magnetic field. What plays the role of the energy eigenvalue in this equation?

(d) We can define H_{c2} to be the largest magnetic field for which the equation you found in (c) has a well-behaved solution, i.e., a solution for which $|\psi(\mathbf{r})|$ does not diverge as $r \to \infty$. (You should justify this condition, as well as the use of the linearized form of (1)). Using what you know about the energy levels of a charged particle in a magnetic field show that

$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2}.$$

For what value of the Ginzburg-Landau parameter κ does $H_{c2} = H_c$?

5.3 Consider a Josephson junction of finite width L with a uniform magnetic field in the insulating oxide layer, as shown below. In the upper superconductor, beyond the penetration depth which for simplicity we take here to be zero, the Ginzburg-Landau order parameter can be written $\psi_+(x) = \psi_0 e^{i(\alpha x+\phi_+)}$ where ψ_0 and ϕ_+ are constant. Likewise, in the lower superconductor the Ginzburg-Landau order parameter can be written $\psi_-(x) = \psi_0 e^{i(-\alpha x+\phi_-)}$.



(a) Determine α in terms of the magnetic field H. In the line integrals you will have to consider to do this you may neglect the contribution of the path parallel to z at the ends.

(b) Determine the maximum DC supercurrent $J_1(H)$ which the junction can pass as a function of the field H. You will need to define a Josephson parameter per unit length j_1 such that

$$J_1(0) = \int_{-L/2}^{L/2} j_1 dx = Lj_1$$

and the current *density* at any point in the junction is proportional to j_1 times the sine of the phase difference between ψ_+ and ψ_- at that point.

(c) Write $J_1(H)$ as a function of the number (which need not be integer) of flux quanta in the junction and sketch the result. Is there an optical analog of this effect?

(Note: In a real junction, the penetration depth will ordinarily be much greater than the oxide thickness and t should be replaced by a distance of the order of twice the penetration depth.)